A Mortar Mimetic Finite Difference Method on Non-Matching Grids

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We consider mimetic finite difference approximations to the mixed form of second order elliptic problems on non-matching quadrilateral and triangular multi-block grids. Mortar finite elements are employed on the non-matching interfaces to impose weak continuity of the velocity. Optimal convergence and, for certain cases, superconvergence is established for both the pressure and the velocity.

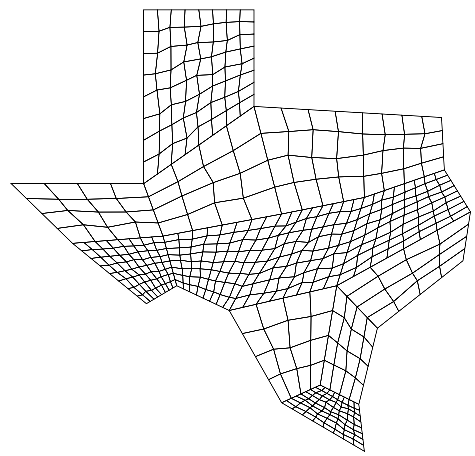
Background

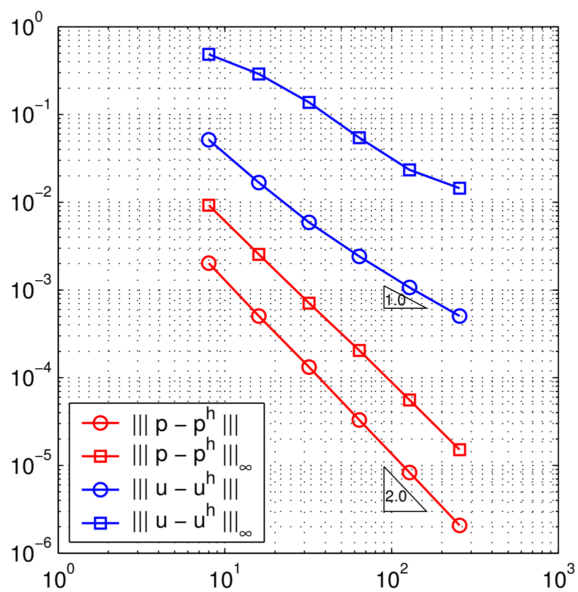
In this work, we consider second order linear elliptic equations that in porous medium applications model single phase Darcy flow. We solve for pressure p and velocity $\mathbf{u}$ satisfying

$$\begin{aligned} \mathbf{u} &= -K\nabla p && \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= b && \text{in } \Omega, \\ p &= g && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$, is a multi-block domain, and K is a symmetric, uniformly positive definite tensor with $L^\infty(\Omega)$ components representing the rock permeability divided by the fluid viscosity. The Dirichlet boundary conditions are considered merely for simplicity.

The mortar mixed finite element (mortar MFE) method has been studied, for example, in [1, 2]. In this method, the domain is divided into nonoverlapping subdomain blocks, and each of these subdomain blocks is discretized on a locally constructed mesh. As a result, the subdomain grids do not match at interblock boundaries. To solve this problem, Lagrange multiplier pressures are introduced at the interblock boundaries. This Lagrange multiplier space is called the mortar finite element space. It was shown in [2] that the mortar MFE method is optimally convergent, if the boundary space has one order higher approximability than the normal trace of the velocity space.





A randomly perturbed multi-block quadrilateral mesh (top) and a convergence study on a sequence of such meshes using our mortar finite difference method (bottom).

A connection between the mimetic finite difference (MFD) method and the MFE method with Raviart-Thomas finite elements was established in [3]. This was achieved by showing that the scalar product in the velocity space proposed in [4] for MFD methods can be viewed as a quadrature rule in the context of MFE methods. In [5], superconvergence for the normal velocities in MFD methods on h^2-uniform quadrilateral meshes is established.

Our Main Results

The MFD scalar product is not unique. In [6], we analyze two inner products. Both of them are numerically much cheaper than the inner product that is used in MFE methods. (For example, a complication with MFE methods on quadrilateral meshes is that the Piola transform requires the integration of rational functions.)

In [6], we show optimal convergence of the pressure and the velocity in the mortar MFD method in the L^2 norm ($O(h)$). Under stronger assumptions on the grid (h^2-uniform quadrilateral), we show superconvergence of the velocity ($O(h^{3/2})$) in a discrete norm. For the pressure, we show a superconvergence result ($O(h^2)$) for grids consisting of simplicial elements or h^2-parallelograms. The same estimate holds also for a discrete norm, even on general quadrilaterals.

Numerical Experiments

The graph in the figure shows convergence rates for the pressure and velocity on a sequence of randomly perturbed multi-block quadrilateral meshes of the type that is depicted above it. The mortar MFD method with our special scalar product has been used on this random mesh. The asymptotically optimal convergence rate for the velocity is in agreement with our theory. The second order convergence rate for the pressure variable was also theoretically predicted.

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